Imperial College London

Lecture 7

Capacitors & Inductors

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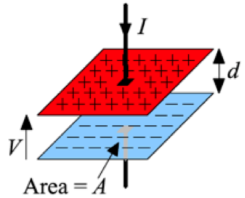
In addition to resistors that we have considered to date, there are two other basic electronic components that can be found everywhere: the capacitor and the inductor. We will consider these two types of components in this lecture and also in Lab Experiment 2.

Capacitors & Capacitance

- A capacitor is formed from two conducting plates separated by a thin insulating layer called a **dielectric**.
- If a current i flows, positive change, q, will accumulate on the upper plate. To preserve charge neutrality, a balancing negative charge will be present on the lower plate.



- There will be a potential energy difference (or voltage v) between the plates proportional to q.

$$v = \frac{d}{A\epsilon} q$$

where A is the area of the plates, d is their separation and ε is the permittivity of the insulating layer ($\varepsilon_0 = 8.85\ pF/m$ for a vacuum).

- The quantity $C = A\varepsilon/d$ is the capacitance and is measured in Farads (F). Hence $q = Cv$, and the current i is the rate of charge on the plate.

The Capacitor Equations:

$$q = Cv, \text{ therefore } \frac{dq}{dt} = i = C\frac{dv}{dt} \text{ and } v = \frac{1}{C}\int i\,dt$$

P326-327

Besides resistors, capacitors are one of the most common electronic components that you will encounter. Sometimes capacitors are components that one would deliberately add to a circuit. Other times, capacitors are side effects that come about even if we don't want them.

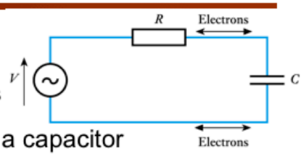
The simplest capacitor is formed by an insulating material (known as dielectric) sandwiched between two parallel conducting plates. When a voltage potential is applied to the two ends, charge accumulates on the plates.

In capacitors, voltage *v* is proportional to the charged stored q. The constant of proportionality is the capacitance C. Since current is the rate of change of charge (i.e. the flow of charge), the relationship between *v* and *I* involves differentiation or integration.

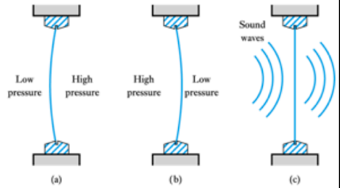
Capacitance is measured in Farads.

DC, AC and Capacitors

- A constant current (DC) cannot flow through a capacitor
  - There is an insulator between the two terminals



- An alternating current (AC) can "flow through" a capacitor
  - Since the voltage across a capacitor is proportional to the charge on it, an alternating voltage must correspond to an alternating charge
  - This can give the impression that an alternating current flows through the capacitor
- A mechanical analogy:
  - Air (charge) cannot pass through a window in spite of the pressure difference (voltage potential)
  - However, alternating pressure can make the window vibrates, produces air movement



Since there is an insulating layer between the two conducting plates of a capacitor, DC current cannot flow through a capacitor. So always remember: **A CAPACITOR IN SERIES BLOCKS DC part of a signal**. However, alternating or changing current can flow through a capacitor. The best analogy is the flow of air from inside to outside of the building. Assuming that the window is completely sealed, air inside the building cannot flow to the outside in spite of the pressure difference between the two sides. The pressure difference is analogous to the voltage potential at the two end of the capacitor. The air flow is like DC current.

However, if the air pressure difference is alternating, there can be air movement on both sides as shown in the diagram.

Capacitors in Series and in Parallel

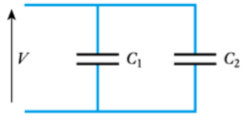
- **Capacitors in parallel**
  - consider a voltage *V* applied across two capacitors
  - then the charge on each is

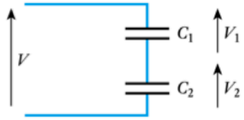
    $Q_1 = VC_1$ and $Q_2 = VC_2$
  - if the two capacitors are replaced with a single capacitor C which has a similar effect as the pair, then

    Charge stored on combined $C = Q_1 + Q_2$

    $\Rightarrow VC = VC_1 + VC_2$

    $\Rightarrow C = C_1 + C_2$





- **Capacitors in series**
  - consider a voltage *V* applied across two capacitors in series
  - the only charge that can be applied to the lower plate of C_1 is that supplied by the upper plate of C_2. Therefore the charge on each capacitor must be identical.
  - Let this be Q, and therefore if a single capacitor C has the same effect as the pair, then:

    $V = V_1 + V_2 \Rightarrow Q/C = Q/C_1 + Q/C_2$

    $\Rightarrow 1/C = 1/C_1 + 1/C_2$

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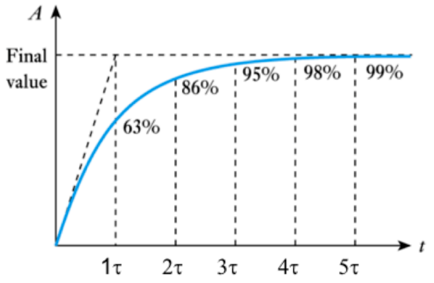
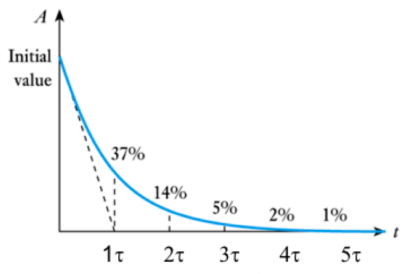
Connecting two capacitors in parallel results in their capacitances ADD-ed together (just like resistors in series).

Connecting two capacitors in series results in their capacitances combined in a product/sum manner, similar to two parallel resistances.

The Exponential Signal



$$A(1 - e^{-t/\tau})$$



$$Ae^{-t/\tau}$$

P327

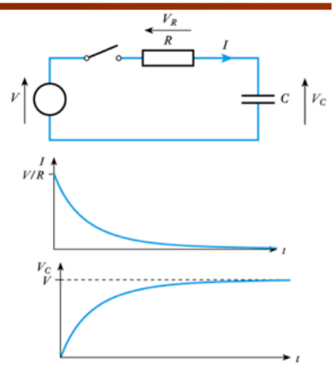
Before we embark on circuits using capacitors, let us examine one of the signals that you explored in Lab 1 in the past two weeks – the exponential signal.

Exponential signals are interesting. Here the rate of change is shown in terms of the time constant t (tau).

The following facts are worth remembering:

1. For exponential rise, the signal reaches 63% at one t, and 95% at 3t.
2. For exponential fall, the signal reaches 37% at one t, and 5% at 3t.

Capacitor and the Exponential



- Consider the circuit shown here
  - capacitor is initially discharged
    - voltage across it will be zero
- Switch is closed at t = 0
  - V_C is initially zero
    - V_R is initially V, I is initially V/R
- As the capacitor charges:
    - V_C increases, V_R decreases
    - I decreases
    - We have exponential behaviour in both I and V_C
- **Time constant**
  - Charging current *I* is determined by *R* and the voltage across it
  - Increasing *R* will increase the time taken to charge *C*
  - Increasing *C* will also increase time taken to charge *C*
  - Time required to charge to a particular voltage is determined by ***CR***
  - This product ***CR*** is the **time constant** τ (greek tau)

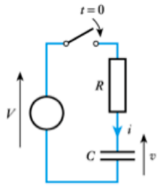
For the circuit shown here, assume the capacitor has zero charge (and 0v) at t = 0. The switch is closed, connecting the circuit to the constant voltage source Vs. Initially the voltage drop across the resistor is Vs. A current of Vs/R flows from the source to capacitor. However, a Vc increases, the current I decreases. This results in the exponential drop of changing current and an exponential rise of the capacitor voltage. We will examine mathematically how i and Vc changes over time in the next lecture.

For now, it is important to consider the parameter known as the time constant. If R is large, the charging current I is small, and it takes longer to charge the capacitor. For a given R, if C is large, it can store more charge for a given voltage, therefore the time needed to charge a capacitor to a certain voltage is proportional to the produce R x C. RC is known as the time constant of this circuit.

Step Response of a RC circuit

- Consider what happens to the circuit shown here as the switch is closed at t = 0.
- Apply KVL around the loop, we get:

$$iR+v=V, \ \ but \ \ i=C\frac{dv}{dt} \ \ therefore \ \ RC\frac{dv}{dt}+v=V$$

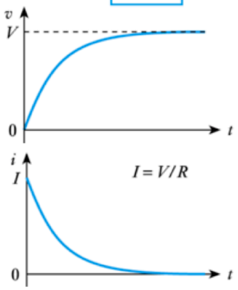


- This is a simple **first-order differential equation** with constant coefficients.
- Assuming V_C = 0 at t = 0, the solution to this is

$$v=V(1-e^{-\frac{t}{RC}})=V(1-e^{-\frac{t}{\tau}}), \ \ \text{where } \tau = RC = \text{time-constant}$$

- Since $i=C\frac{dv}{dt}$ this gives (assuming V_C = 0 at t = 0):

$$i=I\times e^{-\frac{t}{RC}}=I\times e^{-\frac{t}{\tau}}, \ \ \text{where } I=\frac{V}{R}$$



We have seen this circuit before in the last lecture. We will now derive the exponential equation formally. For that you need to be familiar with solving first-order differential equations from your maths lectures.

We want to solve: $RC\frac{dv}{dt}+v=V$

$$\frac{dv}{dt}=\frac{V-v}{RC}$$

$$\Rightarrow \frac{dt}{RC}=\frac{dv}{V-v}$$

Integrate both sides, we get:

$$\frac{t}{RC}=-\ln(V-v)+A, \ \ \text{where A constant of integration}$$

Use boundary condition: when t= 0, v = 0:

$$\frac{0}{RC}=-\ln(V-0)+A$$

$$\Rightarrow A=\ln V$$

Therefore

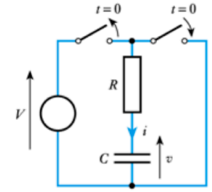
$$\frac{t}{RC}=-\ln(V-v)+\ln V=\ln\frac{V}{V-v}$$

$$\Rightarrow e^{\frac{t}{RC}}=\frac{V}{V-v}\Rightarrow v=V(1-e^{-\frac{t}{RC}})$$

Discharging Capacitor in a RC circuit

- Consider what happens to the circuit shown here as the left switch is open and the right switch closed at t = 0.
- At t = 0, V_C = V.
- Apply KVL around the right loop, we get:

$$iR+v=0, \ \ and \ \ i=C\frac{dv}{dt} \ \ therefore \ \ RC\frac{dv}{dt}+v=0$$

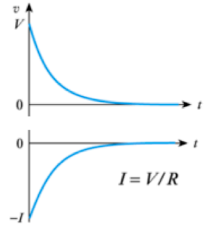


- Solving this simple first-order differential equation gives:

$$v=V\times e^{-\frac{t}{RC}}=V\times e^{-\frac{t}{\tau}}, \ \ \text{where } \tau = RC = \text{time-constant}$$

- And:

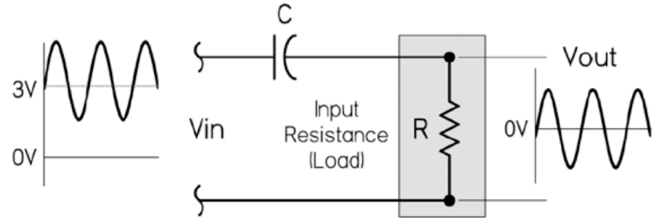
$$i=-I\times e^{-\frac{t}{RC}}=-I\times e^{-\frac{t}{\tau}}, \ \ \text{where } I=\frac{V}{R}$$



Let us now consider what happens if we charge up the capacitor, then at t = 0, discharges it. The equations is also easy to solve and it is clear that the discharge profiles in V and I also follow the exponential curves.

DC Blocking using Capacitor

- Capacitor is often used to present dc voltage from passing from one side of the circuit to another.
- Here, on the left side, the signal has a 3V DC component, and a sinewave superimpose.
- On the right side, the output signal Vout is centred around 0V. That is, the DC input is "blocked" or isolated from the output.
- This use of capacitor is also known as "**AC coupling**".



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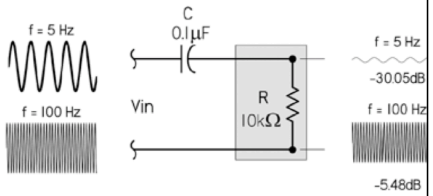
We will now consider some common use of capacitors in electronic circuit. The first application of a capacitor is to remove or "block" DC part of an electrical signal.

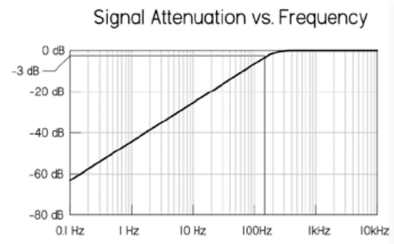
Consider the circuit above. The input signal Vin has a 3V DC voltage on which is a sinewave. By choosing the correct value for C and R, you can obtain an output signal, Vout, which is has 0V DC (i.e. DC is blocked) and only the sinusoidal signal remains.

The capacitor is acting like a window of an airplane. The constant pressure on outside the airplane (DC value) is not affecting the pressure inside the cabin. However, vibration of the window (sinusoidal component) can pass through to the cabin if the window is sufficient flexible relative to the "resistance" of the cabin air.

Filtering effect of Capacitor

- Such circuit also has different effect on the input signal at different frequencies.
- Shown here is two signals, one at 5Hz and another at 100Hz and the C and R values are as given.
- The 5Hz sinewave is suppressed by -30dB or reduced by a factor of 32.
- The 100Hz signal is only reduced by -5.48dB or reduced by a factor of 1.9.
- Therefore, a C in series with a R as shown will give us a high pass filter: a circuit that passes high frequency signals but suppresses low frequency.





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Let us take another view of this circuit besides its DC blocking (or AC coupling) quality.

If C= 0.1uF and R = 10kΩ, you will find that the circuit will strongly suppress a 5Hz sinewave, but if the input signal has a much higher frequency (say 100 Hz), the signal is passed through the capacitor with little or no reduction.

If you plot signal gain (or attenuation), i.e. Vout/Vin, vs Frequency, you can a characteristic as shown in the slide. The low the Y-axis value, the lower the gain and stronger the suppression. In fact if you project the plot toward frequency = 0 (that is, to DC), the gain is –infinity. That's why this circuit can "BLOCK" DC signal.

Note that the plot here has both axes in logarithmic scale. The frequency axis is clearly log scale. How about the y-axis? It is plotting the gain of the circuit in dB or decibel. dB is also in log scale as will be shown in the next slide.

Decibel

- Ratio of output to input voltage in an electronic system is called voltage **gain**:

$$A = \left|\frac{V_{out}}{V_{in}}\right|$$

- If the gain is low than 1, we also call this attenuation.
- Voltage gain of a circuit is often expressed in logarithmic form:

$$A_{dB} = 20\log(\left|\frac{V_{out}}{V_{in}}\right|) = 20\log A$$

- Power gain of a circuit is the ratio of output power to input power, and is also often expressed in dB, but the equation is different:

$$\text{Power Gain in dB} \quad G_{dB} = 10\log\left(\frac{P_{in}}{P_{out}}\right) = 10\log(\frac{{V_{out}}^2}{{V_{in}}^2})$$

$$= 20\log(\left|\frac{V_{out}}{V_{in}}\right|) = 20\log A$$

P204-205

In electronics, we often ask the question: what is the ratio of the output to input signal? This ratio is important. If it is larger than 1, the electronic circuit provides gain (we also call this amplification as will be seen in a later lecture).

If the ratio is lower than 1, then the circuit provides attenuation (or suppression).

However, we often express this ratio or gain not just as such a ratio, but in logarithmic scale. Why? It turns out that expressing such ratio in log scale provide us with much higher **dynamic range**. For example, our human perception is generally in log scale, not linear scale. Our hearing and seeing sensitivity is not linear, but logarithmic.

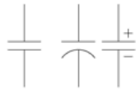
Log - base 10 of the ratio is known as a bel. Scaling this further up by a factor of 10 is known as decibel. That ratio is generally considering the ratio of output power to input power (not voltage). However, since power is voltage square, we found that the common equation to find voltage gain (ratio) in decibel is given by the equation:

$$A_{dB} = 20\log(\left|\frac{V_{out}}{V_{in}}\right|) = 20\log A$$

This is an equation that you must commit to memory – very useful for many things!

Types of Capacitors

- Capacitor symbol represents the two separated plates. Capacitor types are distinguished by the material used as the insulator.



- **Polystyrene**: Two sheets of foil separated by a thin plastic film and rolled up to save space. Values: 10 pF to 1 nF.

- **Ceramic**: Alternate layers of metal and ceramic (a few μm thick). Values: 1 nF to 1 μF.

- **Electrolytic**: Two sheets of aluminium foil separated by paper soaked in conducting electrolyte. The insulator is a thin oxide layer on one of the foils. Values: 1 μF to 10mF.

- Electrolytic capacitors are **polarised**: the foil with the oxide layer must always be at a positive **voltage** relative to the other (else **explosion**).
- Negative terminal indicated by a curved plate in symbol or "-".

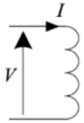
P333-340

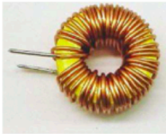
There are many different types of capacitors depending on the method of construction and the materials used. The most common three types are: polystyrene, ceramic and electrolytic. You will choose which to use depending on the operating signal frequency. Polystyrene and ceramic capacitors are good for a wide frequency range, particularly at very high frequencies (say over 1MHz). However, they only come in fairly low capacitance value.

For large capacitances, one would use electrolytic capacitors. Electrolytic capacitors are only good for fairly low frequencies. Furthermore, they have polarity, i.e. it has a positive and a negative terminal. You must connect the capacitor +ve terminal to the more positive voltage than the –ve terminal.

Inductors

- Inductors are formed from coils of wire, often around a steel or ferrite core.





- The magnetic flux within the coil is $\Phi = \frac{\mu N A}{l} i$ where N is the number of turns, A is the cross-sectional area of the coil and l is the length of the coil (around the toroid).
- μ is a property of the material that the core is made from and is called its *permeability*.
- For free space (or air): $\mu_0 = 4\pi \times 10^{-7} = 1.26\,\mu H/m$, for steel, $\mu \approx 4000\,\mu_0 = 5mH/m$.
- From Faraday's law: $v = N\frac{d\Phi}{dt} = \frac{\mu N^2 A}{l}\frac{di}{dt} = L\frac{di}{dt}$.
- We measure the *inductance*, $L = \mu N2A/l$, in Henrys (H).

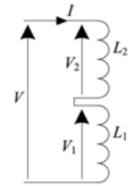
Inductors are the complementary component to the capacitor. They are not commonly found in electronic circuits because they are bulky and expensive, and practical inductors are far from ideal. However, they are found in motors, transformers and other electrical mechanisms. They are also found as stray effects (undesirable side effects) with interconnecting wires (such as wires that you use to connect circuits on the breadboard). Inductors are used as antennae for sending and receiving radio signals, and form part of transformers used in wireless charging.

Here are some basic equations governing an inductor. The most important is v = IL di/dt. Note the similarity to the capacitor equations.

Series and Parallel Inductors

Series inductors:

$$v = v_1 + v_2 = L_1\frac{di}{dt} + L_2\frac{di}{dt} = (L_1 + L_2)\frac{di}{dt}$$

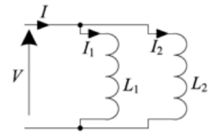


- Same equation as a single inductor of value $L_1 + L_2$

Parallel inductors:

$$\frac{di}{dt} = \frac{d(i_1+i_2)}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = \frac{v}{L_1} + \frac{v}{L_2} = v\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

$$v = \frac{1}{\frac{1}{L_1}+\frac{1}{L_2}}\frac{di}{dt}$$

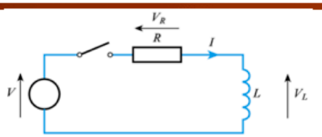


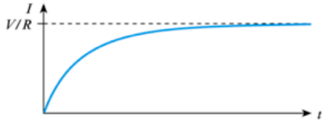
- Same as a single inductor of value $\frac{1}{\frac{1}{L_1}+\frac{1}{L_2}} = \frac{L_1 L_2}{L_1 + L_2}$
- Inductors combine just like resistors.

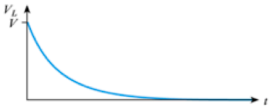
Series and parallel inductors combine just like resistors do.

Inductor and the Exponential

- Consider the circuit shown here
  - Inductor is initially un-energised
    - ➢ current through it will be zero
- Switch is closed at t = 0
  - *I* is initially zero
    - ➢ V_R is initially *0*, V_L is initially *V*
- As the inductor is energised:
    - ➢ *I* increases, V_R increases
    - ➢ V_L decreases
    - ➢ We have exponential behaviour in both *I* and V_L







- **Time constant**
  - In inductor-resistor circuit the time taken for the current to rise to a certain value is determined by the time constant ***L/R***
  - This value ***L/R*** is the **time constant** τ (greek tau)

Energising an inductor is similar to that of charging a capacitor. Except that the inductor CURRENT (as suppose to the capacitor voltage) is rising exponentially. The inductor voltage is decreasing exponentially.

The time constant is L/R as suppose to RC.

Step Response of a LR circuit

- Consider what happens to the circuit shown here as the switch is closed at t = 0.
- Apply KVL around the loop, we get:

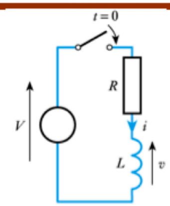
$$iR + v = V, \ but \ v = L\frac{di}{dt} \quad therefore \ iR + L\frac{di}{dt} = V$$

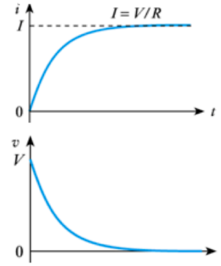
- This is a simple first-order differential equation with constant coefficients.
- Assuming $i_L = 0$ at $t = 0$, the solution to this is

$$i = I(1 - e^{-\frac{t}{L/R}}) = I(1 - e^{-\frac{t}{\tau}}), \quad \text{where } \tau = \frac{L}{R} = \text{time-constant}$$
$$\text{and} \quad I = V/R$$

- Since $v = L\frac{di}{dt}$ this gives (assuming $V_L = 0$ at $t = 0$):

$$v = V \times e^{-\frac{t}{L/R}} = V \times e^{-\frac{t}{\tau}}, \quad \text{where } V = i \times R$$





We can perform the same analysis with an inductor being energised (we don't call this charging). At t=0, when the switch is first closed, NO CURRENT FLOWS, since the current through an inductor cannot change instantaneously. Since no current flows, voltage across the inductor must be V, the same as the voltage source. Therefore as soon as the switch is closed, v goes from 0 to V instantaneously! This is a characteristic of a LR circuit.

The current rises from 0, therefore the voltage drop across the resistor R increases, decreasing the inductor voltage. Solving the first-order differential equation provides the exact equations for i_L and v_L.

De-energising Inductor in a LR circuit

- Consider what happens to the circuit shown here as the left switch is open and the right switch closed at t = 0.
- At $t = 0$, $i_L = V/R$.
- Apply KVL around the right loop, we get:

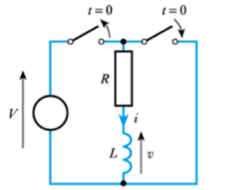
$$iR + v = 0, \quad and \quad v = L\frac{di}{dt} \quad therefore \quad iR + L\frac{di}{dt} = 0$$

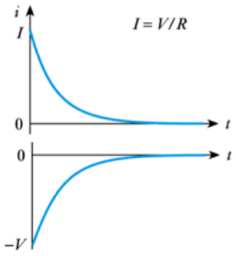
- Solving this simple first-order differential equation gives:

$$i = I \times e^{-\frac{t}{L/R}} = I \times e^{-\frac{t}{\tau}}, \quad \text{where } I = \frac{V}{R} \text{ and } \tau = \frac{L}{R}$$

- And:

$$v = -V \times e^{-\frac{t}{L/R}} = -V \times e^{-\frac{t}{\tau}},$$





Similarly when we de-energise the inductor, we get the exponential characteristics as we did for discharging the capacitor.

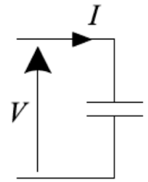
Current / Voltage Continuity

Capacitor: $i = C\, dv/dt$

- For the voltage to change abruptly $dv/dt = \infty \Rightarrow i = \infty$.

This never happens so ...

- **The voltage across a capacitor never changes instantaneously.**
- Informal version: A capacitor "tries" to keep its voltage constant.

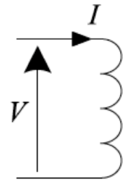


Inductor: $v = L\, di/dt$

- For the current to change abruptly $di/dt = \infty \Rightarrow v = \infty$.

This never happens so ...

- **The current through an inductor never changes instantaneously.**
- Informal version: An inductor "tries" to keep its current constant.



The take-home message that you must remember is that:

Capacitor tries to keep its voltage constant.

Inductor tries to keep its current constant.

Summary

- Capacitor:
  - $i = C\,dv/dt$
  - parallel capacitors add in value
  - v across a capacitor never changes instantaneously.
- Inductor:
  - $v = L\,di/dt$
  - series inductors add in value (like resistors)
  - i through an inductor never changes instantaneously.